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ON BIJECTIONS OF LORENTZ MANIFOLDS, WHICH LEAVE THE CLASS OF SPACELIKE PATHS INVARIANT

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This paper is dedicated to the memory of Dr. Stephen M. Paneitz.

Suppose that (M, g) and (M', g') are Lorentz manifolds, and that $f: M \to M'$ is a bijection, such that f and f^{-1} preserve spacelike paths ($f: M \to M'$ has this property, if for any spacelike path $\gamma: J \to M$ in (M, g), the composition $f\gamma: J \to M'$ is a spacelike path in (M', g')). Then f is a (manifold-) homeomorphism.

This statement is the 'spacelike' version of an analogous 'timelike' theorem (Hawking, King and McCarthy [6] and Göbel [2] for strongly causal, and Malament [10] for general Lorentz manifolds).

With this result it is possible to prove a conjecture of Göbel [3] which states that every bijection between time-orientable n-dimensional ($n \geq 3$) Lorentz manifolds which preserves spacelike paths is a conformal C^∞-diffeomorphism.

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Lorentz manifolds | space topology
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§ 1.

In this paper, a *Lorentz manifold* is a pair (M, g), where M is a connected, Hausdorff, C^∞, real, n-dimensional manifold without boundary (the manifold topology on M will be denoted by $\boldsymbol{M}$), and g is a C^∞ Lorentz metric on M (in every $p \in M$, the canonical form of $g(p)$ is $g_{rs}(p) = \text{diag}(-1, +1, \ldots, +1)$). In the case $n = 4$, (M, g) is called a *space-time*. For all not explicitly defined notions we refer to [5] and [7].

Let (M, g) be a Lorentz manifold. For $p \in M$, $N(p)$ denotes the family of all $\boldsymbol{M}$-open, convex, normal neighbourhoods of p in M. To every $U \in N(p)$ there corresponds a unique open neighbourhood $\tilde{U}$ of zero $0 \in T_pM$, such that $\exp_p | \tilde{U}$ is a C^∞-diffeomorphism of $\tilde{U}$ onto U with $\exp_p(0) = p$. Let $\| \;\; \|$ be any norm in T_pM (which is independent of p). Then $\tilde{B}_p(\varepsilon) = \{X \in T_pM \mid \|X\| < \varepsilon\}$, and for every $U \in N(p)$ there exists an $\varepsilon_U = \sup\{\varepsilon > 0 \mid \tilde{B}_p(\varepsilon) \subseteq \tilde{U}\} > 0$. For every $U \in N(p)$ and

$0 < \varepsilon \leq \varepsilon_U$ we put

$$B_U(p, \varepsilon) = \exp_p \tilde{B}_p(\varepsilon)$$

(notice that $\boldsymbol{N}(p)$ as well as $\{B_U(p, \varepsilon) \mid 0 < \varepsilon \leq \varepsilon_U\}$, for an arbitrary, fixed $U \in \boldsymbol{N}(p)$, are $\boldsymbol{M}$-open neighbourhoodbases in p with respect to $\boldsymbol{M}$). For every $p \in M$,

$$\tilde{R}_p = \{X \in T_pM \mid X = 0 \text{ or } g_p(X, X) > 0\}$$

is called the space-cone in T_pM. For every $U \in \boldsymbol{N}(p)$ and $0 < \varepsilon \leq \varepsilon_U$ we put

$$S(p, U) = \exp_p(\tilde{U} \cap \tilde{R}_p)$$

and

$$R_U(p, \varepsilon) = \exp_p(\tilde{B}_p(\varepsilon) \cap \tilde{R}_p)(= B_U(p, \varepsilon) \cap S(p, U)).$$

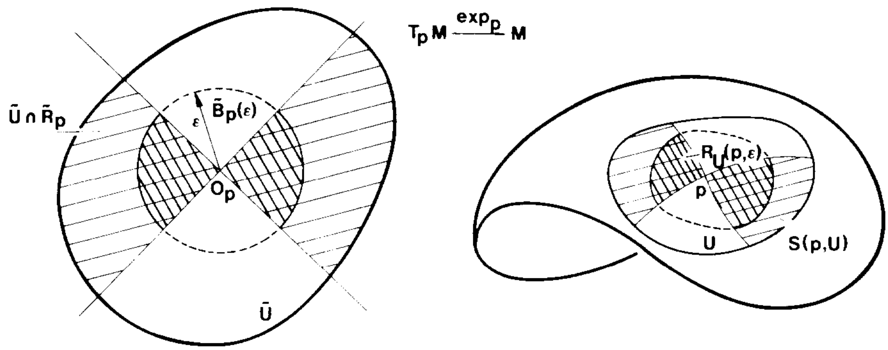

Fig. 1

Let $J \subseteq \mathbb{R}$ be a connected set containing more than one point. $\gamma : J \to M$ is called *spacelike in* (M, g), abbreviated $\gamma \in \Sigma$, if γ is continuous and if for every $t \in J$ there exists an open and connected neighbourhood W of t in J and an $U \in \boldsymbol{N}(\gamma(t))$, such that $\gamma(W) \subseteq S(\gamma(t), U)$. If $\gamma \in \Sigma$ and J is compact, then γ is called a *spacelike path in* (M, g). A subset $C \subseteq M$ is called a *spacelike curve in* (M, g), abbreviated $C \in \boldsymbol{K}$, if there exists a $\gamma : J \to M \in \Sigma$ such that $C = \gamma(J)$ and γ is a homeomorphism of J onto C.

$H \subseteq M$ is called a *spacelike hypersurface* (with or without boundary) *in* (M, g), if H is an $(n-1)$-dimensional C^0-submanifold of M (i.e., the canonical injection $i : H \hookrightarrow M$ is a C^0-imbedding), such that

$$\forall p \in H \; \exists U \in \boldsymbol{N}(p), \;\; H \cap U \subseteq S(p, U).$$

§ 2.

In the proof of our main result, the notion of the space topology for Lorentz manifolds will play a central role. In this section let (M, g) be a fixed Lorentz manifold.

Definition. The finest topology on M with the property, that it induces on every spacelike curve in (M, g) the same topology as the manifold topology is called the *space topology on* (M, g). This topology will be denoted by $\boldsymbol{S}$ (or $\boldsymbol{S}(M, g)$).

Remarks. Following suggestions of Zeeman [12], Nanda [11] introduced the space topology in 1972 in a completely different setting for the Minkowski space-time M_4 (called the 's-topology'). For Lorentz manifolds the space topology was defined in 1976 by Göbel [3] (denoted by $\boldsymbol{X}_{\emptyset/0}$). [1, 3, 8, 9] can be used as general references on this topic.

Finally we put together some facts concerning the space topology which are needed later.

Proposition 1 ([3, Theorem 3.3(a)]). *The space topology $\boldsymbol{S}$ on (M, g) is the finest topology on M, such that for every spacelike hypersurface H in (M, g)*

$$\boldsymbol{S} | H = \boldsymbol{M} | H$$

($\boldsymbol{S} | H$ resp. $\boldsymbol{M} | H$ denotes the subspace topology induced on $H \subseteq M$ by $\boldsymbol{S}$ resp. $\boldsymbol{M}$).

Proposition 2 ([3, Theorem 3.3(c)]). *For every $p \in M$ and $U \in \boldsymbol{N}(p)$, the family*

$$\{R_U(p, \varepsilon) \mid 0 < \varepsilon \leq \varepsilon_U\}$$

is a $\boldsymbol{S}$-open neighbourhoodbase in p with respect to $\boldsymbol{S}$.

Proposition 3. *For $\gamma : J \to M$, $J \subseteq \mathbb{R}$ being a connected set containing more than one point, the following properties are equivalent:*

(i) *γ is $\boldsymbol{S}$-continuous*
(ii) $\gamma \in \Sigma$.

Proposition 1 shows that instead of the spacelike curves we could have used in the definition the spacelike hypersurfaces as well. This seems—from the physical point of view—to be an even more natural starting point to introduce the space topology. Proposition 2 shows that $\boldsymbol{S}$ is strictly finer than $\boldsymbol{M}$, first countable, Hausdorff and separable. Further it can be shown that $\boldsymbol{S}$ is not regular, hence not metrizable, but is semimetrizable [1, Theorem 5]. Finally, Proposition 3 implies that the $\boldsymbol{S}$-paths in M are precisely the spacelike paths in (M, g). Therefore $\boldsymbol{S}$ is arcwise connected and locally arcwise connected.

§ 3.

The main result is:

Theorem 1. *Let (M, g) and (M', g') be two Lorentz manifolds and $f : M \to M'$ a* *bijection, where both f and f^{-1} preserve spacelike paths. Then f is a (manifold-) homeomorphism.*

The proof of this theorem will be subdivided into three parts. $\boldsymbol{K}'$ denotes the spacelike curves in (M', g'), $\boldsymbol{S}' = \boldsymbol{S}(M', g')$, and $\boldsymbol{M}'$ the manifold topology on M'.

Lemma 1. *A bijection $f: M \to M'$ is a $\boldsymbol{S}$-homeomorphism if and only if f and f^{-1} preserve spacelike paths.*

Proof. It is sufficient to show that f is $\boldsymbol{S}$-continuous if and only if f preserves spacelike paths.

$\Rightarrow$. Consequence of Proposition 3.

$\Leftarrow$. It is easy to see that for every $C \in \boldsymbol{K}$ the restriction $f|C: C \to M'$ is $\boldsymbol{M}$-continuous. Next we prove that f is $\boldsymbol{S}$-continuous if $f|C$ is $\boldsymbol{M}$-continuous for every $C \in \boldsymbol{K}$. If we assume that f is not $\boldsymbol{S}$-continuous, then there must exist a $G' \in \boldsymbol{S}'$ such that $Q = f^{-1}(G') \notin \boldsymbol{S}$. By definition of $\boldsymbol{S}$ there exists a $C = \gamma(J) \in \boldsymbol{K}$ such that $C \cap Q \notin \boldsymbol{M}|C$. Therefore we can find an $x \in C \cap Q$ and a sequence (x_n) in $C - Q$, and hence $f(x_n) \notin G'$, with $\lim x_n = x$. Finally, we choose a compact interval $I \subseteq J$ such that $t = \gamma^{-1}(x) \in I$ and $t_n = \gamma^{-1}(x_n) \in I$ for all $n \in \mathbb{N}$. Notice that $f(\gamma|I)$ is a spacelike path in M' and $D = (f\gamma)(I) \in \boldsymbol{K}'$. On the one hand, by the definition of $\boldsymbol{S}'$, we have $f(x) \in D \cap G' \in \boldsymbol{M}'|D$; on the other hand, we have $f(x) = (f\gamma)(t) = \lim(f\gamma)(t_n) = \lim f(x_n)$ in D, and therefore $f(x_n) \in D \cap G' \subseteq G'$ for $n \geqslant N(G') \in \mathbb{N}$. This contradiction disproves the assumption, hence f is $\boldsymbol{S}$-continuous. $\square$

Lemma 2. *Let $f: M \to M'$ be a $\boldsymbol{S}$-homeomorphism and $x \in M$. Then for every $U \in \boldsymbol{N}(x)$ and $0 < \varepsilon \leqslant \varepsilon_U$ we have*

$$f(R_U(x, \varepsilon) - \{x\}) \in \boldsymbol{M}'.$$

Proof. We put $R = R_U(x, \varepsilon)$, and $R_0 = R - \{x\}$. Because of $R_0 \in \boldsymbol{M} \subseteq \boldsymbol{S}$, we have $f(R_0) \in \boldsymbol{S}'$. Assume now that $f(R_0) \notin \boldsymbol{M}'$. Then there exists a $f(y) \in f(R_0)$ and a sequence $(f(y_n))$ in $M' - f(R)$ such that $\boldsymbol{M}' - \lim f(y_n) = f(y)$. Now it can be shown (a proof is given in Appendix A) that for every $p \in M$ and for every $\boldsymbol{M}$-neighbourhood V of p there exists an $\boldsymbol{M}$-compact $\boldsymbol{M}$-neighbourhood X of p (with the $\boldsymbol{M}$-boundary $\partial_{\boldsymbol{M}} X$), which has the following properties:

$$\begin{aligned} &X \subseteq V, \qquad C = \partial_{\boldsymbol{M}} X \text{ is } \boldsymbol{S}\text{-compact}, \\ &A = \operatorname{int}_{\boldsymbol{M}} X \text{ and } B = \operatorname{int}_{\boldsymbol{M}}(M - X) \text{ are } \boldsymbol{S}\text{-separated}. \end{aligned} \tag{$*$}$$

We select such a neighbourhood X for y in R_0 with the properties stated in $(*)$, and a sequence (x_n) in $A - \{y\}$, such that $\boldsymbol{S} - \lim x_n = y$. f is an $\boldsymbol{S}$-homeomorphism, hence $\boldsymbol{S}' - \lim f(x_n) = f(y)$, and $C' = f(C) \subseteq M'$ is an $\boldsymbol{S}$-compact subset. Therefore, C' is $\boldsymbol{M}'$-compact, hence $\boldsymbol{M}'$-closed. Because of $f(y) \notin C'$ there exists a $W' \in \boldsymbol{N}(f(y))$ with $W' \cap C' = \emptyset$. The sequences (x_n) and (y_n) are contained in different $\boldsymbol{S}$-

components of $M-C$. Therefore the sequences $(f(x_n))$ and $(f(y_n))$ must also be contained in different $\boldsymbol{S}'$-components of $M'-C'$. But since $\boldsymbol{M}'-\lim f(x_n)=\boldsymbol{M}'-\lim f(y_n)=f(y)$, we have at the one hand the existence of an index $r\in\mathbb{N}$, such that $f(x_r)\in W'$ and $f(y_r)\in W'$. On the other hand, it can easily be seen that there exists an $\boldsymbol{S}'$-continuous path $\alpha:[0,1]\to M'$ such that $\alpha([0,1])\subseteq W'$, $\alpha(0)=f(x_r)$, and $\alpha(1)=f(y_r)$. This contradiction refutes the assumption $f(R_0)\notin\boldsymbol{M}'$. □

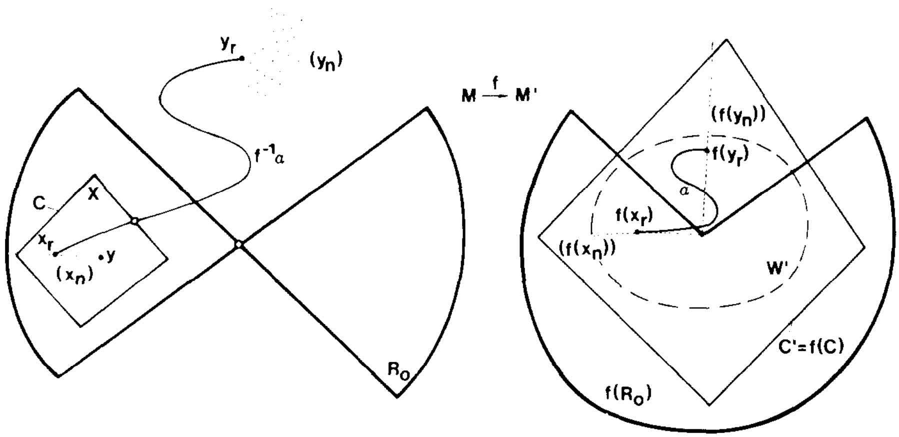

Fig. 2

Lemma 3. *Every* $\boldsymbol{S}$*-homeomorphism is an* $\boldsymbol{M}$*-homeomorphism.*

Proof. Let $f:M\to M'$ be a $\boldsymbol{S}$-homeomorphism and $G\in\boldsymbol{M}$ nonempty. For every $x\in G$ there exists a $U\in\boldsymbol{N}(x)$ such that $U\subseteq G$. Now we can choose a $y\in U-\{x\}$ such that $x\in R_U(y,\varepsilon)-\{y\}\subseteq U\subseteq G$. Then $f(x)\in f(R_U(y,\varepsilon)-\{y\})\subseteq f(G)$ and, because of Lemma 2, $f(R_U(y,\varepsilon)-\{y\})\in\boldsymbol{M}'$. Therefore, $f(G)\in\boldsymbol{M}'$. In the same way we can show that for every $G'\in\boldsymbol{M}'$ we have $f^{-1}(G')\in\boldsymbol{M}$. Hence f is a $\boldsymbol{M}$-homeomorphism. □

Lemma 1 and Lemma 3 prove Theorem 1.

For the formulation and the proof of the next statement we use the notions from [6, Sections 2-4].

Lemma 4. *Let* (M,g) *and* (M',g') *be time-orientable, and* $\boldsymbol{P}$ *(resp.* $\boldsymbol{P}'$*) denote the path topology on* (M,g) *(resp.* (M',g')*). Then every* $\boldsymbol{S}$*-homeomorphism is a* $\boldsymbol{P}$*-homeomorphism.*

Proof. At first we note that the path topology $\boldsymbol{P}$ is first countable [6, Theorem 3] and that $\boldsymbol{P}$ is strictly finer than $\boldsymbol{M}$ [6, Section 4]. Let $f:M\to M'$ be a $\boldsymbol{S}$-homeomorphism, $x\in M$, (x_n) a sequence in M, such that $\boldsymbol{P}$-$\lim x_n=x$, and assume that $(f(x_n))$ does not converge to $f(x)$ with respect to $\boldsymbol{P}'$. Because of Lemma 3 and

$\boldsymbol{M}$-lim $x_n = x$, we have $\boldsymbol{M}' - \lim f(x_n) = f(x)$. By [6, Theorem 1], there exists a $\boldsymbol{P}$-open neighbourhood $L_U(x, \varepsilon)$ of x, a $\boldsymbol{P}'$-open neighbourhood $L_{V'}(f(x), \delta)$ (with $U \in \boldsymbol{N}(x)$, $0 < \varepsilon \leqslant \varepsilon_U$, $V' \in \boldsymbol{N}(f(x))$, and $0 < \delta \leqslant \varepsilon_{V'}$) and a subsequence $(f(x_{n_k}))$ of $(f(x_n))$, such that

(i) $\forall k \in \mathbb{N}$, $x_{n_k} \in L_U(x, \varepsilon)$;

(ii) $\forall k \in \mathbb{N}$, $f(x_{n_k}) \notin L_{V'}(f(x), \delta)$;

(iii) $\boldsymbol{M}' - \lim f(x_{n_k}) = f(x)$.

If we put $T = \{x_{n_k} | k \in \mathbb{N}\}$, then it is not difficult to see that (i) implies on the one hand $T \cap \mathrm{cl}_{\boldsymbol{S}} R_U(x, \varepsilon) = \emptyset$. On the other hand, (ii) and (iii) imply that for every ω, $0 < \omega \leqslant \varepsilon_{V'}$, $f(T) \cap \mathrm{cl}_{\boldsymbol{S}} R_{V'}(f(x), \omega) \neq \emptyset$. Therefore $f(T \cap \mathrm{cl}_{\boldsymbol{S}} R_U(x, \varepsilon)) \neq \emptyset$. This contradiction disproves the assumption, hence f is $\boldsymbol{P}$-continuous. Analogously it can be shown that f^{-1} is $\boldsymbol{P}'$-continuous. □

Lemma 4 and [10, Theorem 3] give immediately:

Theorem 2. *Let (M, g) and (M', g') be time-orientable, n-dimensional ($n \geqslant 3$) Lorentz manifolds, $\boldsymbol{S} = \boldsymbol{S}(M, g)$, and $\boldsymbol{S}' = \boldsymbol{S}(M', g')$. Then every* **S***-homeomorphism is a conformal C^∞-diffeomorphism.*

Theorem 2 answers in the positive a conjecture of Göbel which was known to be true for strongly causal Lorentz manifolds only [3, Theorem 4.2 (1.b)].

Appendix A. The proof of (*)

Let (M, g) be a fixed Lorentz manifold of dimension n and $M_d = (\mathbb{R}^d, h)$ the d-dimensional Minkowski space (h denotes the Lorentz metric $h = h(p) = (h_{rs}(p)) = \mathrm{diag}(-1, +1, \ldots, +1)$ and $\boldsymbol{M}_d$ denotes the Euclidean topology in M_d).

A subset $Y \subseteq M$ is called *spacelike* if

$$\forall y \in Y\ \exists U \in \boldsymbol{N}(y),\quad Y \cap U \subseteq S(y, U).$$

This definition generalizes the notion of spacelike hypersurfaces to arbitrary subsets. It is easy to see, that every subset of a spacelike subset is again spacelike, and that $Y \subseteq M$ is spacelike if and only if $\boldsymbol{M} | Y = \boldsymbol{S} | Y$.

By a fundamental theorem of Greene [4, Theorem 8], every n-dimensional Lorentz manifold (M, g), locally, can be C^∞ and isometrically embedded in the Minkowski space M_d, with $d = \frac{1}{2}n(n+3)$. As an immediate consequence of this theorem, for every $p \in M$ and every $\boldsymbol{M}$-neighbourhood U of p there exist V, $W_0 \in \boldsymbol{N}(p)$, such that

(1) $W = \mathrm{cl}_{\boldsymbol{M}} W_0 \subsetneqq V \subseteq U$,

(2) W is $\boldsymbol{M}$-compact,

(3) $F: V \to M_d$ is a C^∞-embedding and isometry into M_d.

V is an open submanifold of M, hence $V^* = F(V)$ can be considered as a Lorentz submanifold of M_d which is C^∞ and isometrically embedded in M_d. We put

$W_0^* = F(W_0)$, $W^* = F(W)$, and $p^* = F(p)$, and note that $p^* \in W_0^* \subseteq W^* = \mathrm{cl}_{\boldsymbol{M}_d} W_0^* \subseteq V^*$ and that W^* is $\boldsymbol{M}_d$-compact.

Now we choose a d-dimensional compact simplex Q^* in M_d, such that

(4) $p^* \in \mathrm{int}_{\boldsymbol{M}_d} Q^*$,

(5) $X^* = Q^* \cap V^* \subseteq W_0^*$,

(6) $\partial_{\boldsymbol{M}_d} Q^*$ is spacelike.

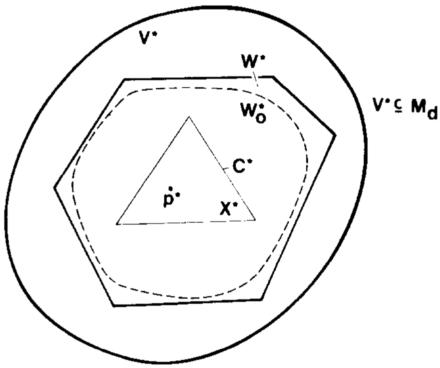

Fig. 3

Now we denote by $\boldsymbol{W} = \boldsymbol{M}_d \mid W^*$ the subspace topology induced by $\boldsymbol{M}_d$ on W^*. (4) and (5) imply

(7) $p^* \in \mathrm{int}_{\boldsymbol{W}} X^*$;

(2) and (5) imply

(8) X^* is $\boldsymbol{W}$-compact.

By an easy calculation we get

(9) $C^* = \partial_{\boldsymbol{W}} X^* \subseteq \partial_{\boldsymbol{M}_d} Q^*$.

(6), (8) and (9) imply

(10) C^* is spacelike and $\boldsymbol{W}$-compact.

By (3), $F^{-1}: V^* \to V$ is a C^∞-diffeomorphism and isometry. If we put

$$X = F^{-1}(X^*),$$

then by (1), (5) and (7) we get

$$p \in \mathrm{int}_{\boldsymbol{M}} X \subseteq X \subseteq V,$$

and by (8), that X is $\boldsymbol{M}$-compact. (10) implies that $C = F^{-1}(C^*) = \partial_{\boldsymbol{M}} X$ is spacelike and $\boldsymbol{M}$-compact. Therefore, because of $\boldsymbol{M} \mid C = \boldsymbol{S} \mid C$, we get that C is $\boldsymbol{S}$-compact. This proves the first two statements in (*).

Finally we put

$$A = \mathrm{int}_{\boldsymbol{M}} X, \qquad B = \mathrm{int}_{\boldsymbol{M}}(M - X).$$

Trivially, $A \neq \emptyset$ and $B \neq \emptyset$. Because of $A \cap \mathrm{cl}_M B = B \cap \mathrm{cl}_M A = \emptyset$, $\mathrm{cl}_S A \subseteq \mathrm{cl}_M A$, and $\mathrm{cl}_S B \subseteq \mathrm{cl}_M B$ we get

$$A \cap \mathrm{cl}_S B = B \cap \mathrm{cl}_S A = \emptyset,$$

i.e., A and B are S-separated. This proves the third statement in $(*)$.

References

[1] R.Z. Domiaty, Topologies for Lorentz manifolds: The space-topology, General Relativity and Gravitation (in print).
[2] R. Göbel, The smooth-path topology for curved space-time which incorporates the conformal structure and analytic Feynman tracks, J. Math. Phys. 17 (1976) 845-853.
[3] R. Göbel, Natural topologies on Lorentzian manifolds, Mitteilungen der Mathematischen Gesellschaft in Hamburg 10 (1980) 763-771.
[4] R.E. Greene, Isometric embeddings of Riemannian and pseudo-Riemannian manifolds, Memoirs AMS 97 (1970).
[5] S.W. Hawking and G.F.R. Ellis, The Large Scale Structure of Space-time (Cambridge University Press, Cambridge, 1973).
[6] S.W. Hawking, A.R. King and P.J. McCarthy, A new topology for curved space-time which incorporates the causal, differential, and conformal structures, J. Math. Phys. 17 (1976) 174-181.
[7] N.J. Hicks, Notes on Differential Geometry (Van Nostrand, New York, 1971).
[8] O. Laback, Topologies in general space-time, Bericht Nr. 138, Forschungszentrum Graz, Austria, 1980.
[9] U. Lindström, Zeeman topologies for space-time, Bericht Nr. 116, Forschungszentrum Graz, Austria, 1979.
[10] D.B. Malament, The class of continuous timelike curves determines the topology of spacetime, J. Math. Phys. 18 (1977) 1399-1404.
[11] S. Nanda, Weaker versions of Zeeman's conjectures on topologies for Minkowski space, J. Math. Phys. 13 (1972) 12-15.
[12] E.C. Zeeman, The topology of Minkowski space, Topology 6 (1967) 161-170.